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SEVENTEENTH YEAR PART SECOND

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# A REVIEW OF THE CHEMICAL CONSTITUTION OF TOURMALINE AS INTERPRETED BY PENFIELD, FOOTE AND CLARKE.

JOSEPH HYDE PRATT.

During the past ten years there have appeared three articles relating to the chemical composition of tourmaline; one on the Chemical Composition of Tourmaline by Prof. S. I. Penfield and H. W. Foote of Yale University, another by Prof. F. W. Clarke of Washington, on the Chemical Constitution of Tourmaline, and a third by Prof. G. Tschermak on "Uber das Mischungsgesetz der Tourmaline." The views presented in these three papers all differ from each other and the latter two lead to very complicated formulae for tourmaline.

The paper by Prof. Penfield and Dr. Foote which was prior to the other two, takes up at length and in detail analyses of tourmaline from DeKalb, N. Y. and Hadden Neck,

Conn., giving method of selection and preparation of material, method of analysis and constitution of tourmaline. This paper also reviews very carefully the old analyses of tourmaline and in many cases shows the relation of these analyses to those that they had made. When it is considered that the two analyses referred to above on the DeKalb and Hadden Neck tourmaline, were made under the direction of the acknowledged foremost chemo-mineralogist of the world and who is known to be the most pains-taking and accurate in all chemical work he does, the conclusions that are deduced from these analyes made by him or under his direction, must have the greatest weight, even if they are to a certain extent anatagonistic to results obtained from analyses made years before. The analyses of minerals which were made prior to 1870 were conducted under difficulties that have been very much lessened since that time. It is now possible to obtain pure material for analysis by the aid of heavy-solutions that are now at hand where formerly that analyzed was often acknowledged to be impure. Also in the advancement of Chemistry, new and improved methods have been devised by which it is now possible to make satisfactory and accurate determinations of various elements contained in the different minerals, where formerly great difficulty was experienced in determining certain of these and in some cases they were not determihed at all.

It is observed that in many of the older mineral analyses no mention is made of the method of analysis or of the purity of the material analyzed and sometimes only an average of two or more analyses is given, so that those working at the present time on these or similar minerals have no way of making accurate comparisons between their results and those formerly obtained, and also there is no way of determining any error that might have been made in the old analyses.

Penfield and Foote give, as the result of their work, as the formula for tourmaline, $H_{20}B_2Si_4O_{21}$, or $H_{18}(B.OH)SIO_{19}$. In connection with the earlier analyses of tourmaline, they

have determined the oxygen ratios in thirty-three of these, from various localities of the world and in nearly all cases they approximate the formula derived by them and some instances it is very close to this. It must, however, be remembered that these earlier analyses were often made upon impure muterial and by methods that have been proved to be inaccurate. The formula they have deduced for tourmaline while it reprents a complex boro-silic acid is a simple formula and one to which all tourmalines can be referred. That the tourmaline contains varying proportions of metals having different valences and of essentially different character, which replace the different hydrogens of the acids may seem peculiar, but while we cannot prove definitely regarding the molecular structure of minerals, still we can get some light on this subject from the various elaborate studies that have been made in organic chemistry. These have shown that one, two, three or more hydrogens can be replaced by simple metallic radicals or simple or compouud organic radicals. If this is true in organic chemistry, it is just as true in inorganic chemistry and while as we said before, this cannot be proved conclusively as it can in organic chemistry still we have every right to believe that this is true and also to go a step futher and make the statement that the differen thydrogen atoms can be replaced by univalent, bivalent, or trivalent metals.

Prof. Clarke is taking for the formula of the tourmaline acid $H_{29}B_3Si_6O_{31}$ There are but few, however of the analyses that have been made of tourmaline, that correspond to this formula. In order to find formulae that will yield calculated percentage values that will agree with the different analyses, Clarke has introduced two formulae which he designates as $A = H_{29}B_3Si_6O_{31}$, $B = H_{31}B_3Si_6O_{32}$, and refers to the various tourmalines as being made up of different proportions of these two molecules. It will be seen that there is but little difficulty in making any of the tourmaline analyses correspond to formulae made up of combinations of these two molecules with different bases, but it leads to a molecule of

tourmaline that is made up of a great number of atoms, up to 3499, which has been calculated for a black tourmaline from Auburn, Maine. Prof. Clarke has given apparently as much weight to the older analyses as he has to the very recent ones. It will be found, however, that careful analytical work of the present time made by pains-taking analysts upon material that has been purified as far as possible and using analytical methods that are known to be accurate, that the formula deduced from these ana lyses, will be found to be identical in most cases with the formula deduced from analyses made of the corresponding mineral from other localities. Also it is more reasonable to accept a simple formula than a more complete one, for t is a more natural supposition that the atoms are combining with each other in simple ratios to form the various mineral molecules, than in complex ratios.

Tschermak takes two formulae $Si_{12}B_6Al_{16}Na_4H_8O_{63}=(Tu)$, and $Si_{18}B_6Al_{20}Mg_{12}H_6O_{63}=(Tm)$, to which he refers all the tourmalines. Those formulae are derived from an an acid $H_{60}B_6Si_{12}O_{63}$ which it will be seen is three times the formula $H_{20}B_2Si_4O_{21}$ which was deduced by Penfield and Foote. The question at once arises why Tschermak took the formula $H_{60}B_6Si_{12}O_{63}$ rather than one third this, the simpler one; and the one of Penfield and Foote. There is no argument in favor of this, Tschermak simply stating that this has been proved to be the acicd formula for tourmalin·. We can determine by substitution the formulae of many organic compounps whether they are the simpler formula or a multiple of it, but with our minerals this can not be determined and the simplest multiple of a formula has always been selected to represent the mineral. To carry out his theory Tschermak is obliged to take a third formula $S_{12}B_6Al_6Mg_{22}P_6O_{63}=(Tn)$ to make the composition of some of the tourmaline correspond to his proposed theory. He as well as Clarke has used all the old analyses, apparently considering them all accurate and unquestionable in the determination of the formulae advanced by them. Penfield on the other hand has derived from analyses that he knows to be accurate and to have been made upon the purest kind of

material; also he has investigated many of the older analyses, and proved conclusively in a number of cases that the material used was impure and that the methods employed would not give accurate results. Tschermak and Clarke are carrying their idea of the existence of mica molecules into the discussion of the constitution of tourmaline and try to make the analyses of tourmaline correspond to this same idea. There is, however, no tangible reason why one should suppose there is any relation between mica and tourmaline. Muscovite has been found pseudomorph after many minerals, some of which, as corondrum, have no relation whatever to it.

Prof. Penfield in a later article, A Criticism on the Constitution of the Tourmaline, discusses the reasons advanced for his formula for tourmaline, and points out certain fallacies that are apparent in Prof. Clark's and Prof. Tschermak's discussions for the acceptance of their respective formulae. It has been generally conceded that one or more analyses of a mineral made at the present time by accurate workers under the most favorable circumstances are of much more value and and infinitely more reliable in determining the chemical constitution or formula of the mineral, than *all* the earlier analyses. Also instead of taking some formula approximately derived from these earlier analyses and attempting to make the recent and known to be accurate analyses correspond to it, how much more reasonable is it to accept the formula derived directly from these recent analyses and to show how the earlier ones correspond to it closely in practically all cases, and sometimes identical with it.

Since Penfield's formula is the simplest, and is derived directly from recent analyses and is one to which all tourmalines can unquestionably be referred while those of Clarke and Tschermak are more complex and in most cases requlre a pair and sometimes three formmulae in order to make the various tourmalines correspond to them, is not the weight of the argument in favor of the acceptance of the Penfield-Foote formula in preference to either of the latter ones?

# SOME EAST AMERICAN SPECIES OF CRATAEGUS.

CONTRIBUTIONS FROM MY HERBARIUM, NO. XI.

W. W. ASHE.

CRATAEGUS RAVIDA. A small tree, 3-5m. in height, with short horizontal branches forming an oblong or oval crown, and dark brown scaly bark on the slender trunk; or sometimes a large schrub. Twig of the season stout, glabrous, red-brown becoming dull gray the second year, marked with a few pale round or oblong lenticels, armed with numerous stout 4-5cm. long dark purlish or blackish thorns; winter-buds ovate or subglobose, bright red-brown. Leaves glabrous, thick and firm, blades broadly ovate or nearly orbicular in outline, 4-6 cm. long, 3.5-5.5cm. wide, obtuse or subacute at the apex, rounded or boadly cuneate at the serrulate base, coarsely and doubly glandular serrate with 2-3 pairs of shallow irregular notches, 3-4 pairs of prominent deeply impessed primary veins which usually fork at or above the middle; petiole stout, 1.5-2cm. long, deeply grooved above, roughened with several pairs of red-brown glands, winged at the apex by the decurrent blade. Inflorescence a simple glabrous several-flowered cyme; calyx narrowly obconic, large, the short entire or serrulate lobes broadly triangular: stamens 10, large; styles 3-5, usually 4. The fruit, in simple 2-4 fruited clusters, on spreading glabrous pedicels, 1-2cm. long, is subglobose, somewhat thicker than long, 14-21mm. long, 16-24mm. thick, full and rounded at the apex, concave at base, russet, or olive mottled with orange, or orange-red, or sometimes yellow, capped by the short erect or spreading calyx lobes, their margins involute; flesh thick orange-yellow, firm and juicy, barely sweetish; seed 3-5, gen-

ally 4, large and coarse, 7-10mm. long, deeply grooved or nearly smooth on the broad rounded back, nearly central in the fruit; the cavity cylindrous, about 4mm. wide, very deep. The fruit falls soon after the foliage, late in October or in November, with the short, stout pedicels attached. In Washington County, Tennessee where this plant was observed, especially around Clarkson, it is uncommon. It grows in sunny rocky woods or on roadsides, often associated with the cock-spur thorn.

CRATAEGUS SCHUETTEI. A spreading much branched schrub 2-3m. in height. Twig of the season slender, glabrous, nearly straight, bright brown, becoming dull gray in the second year, armed with slender 4-6cm. long brown-purple thorns. Leaves thin but firm, glabrous beneath, pubescent when young on the upper surface especially on the midrib with short appressed hairs, soon becoming glabrous; the blades ovate or broadly ovate in outline, 3-6cm. long, 2.5-5cm. wide, acute at the apex, rounded or obtuse at base, very sharply and doubly glandular serrate except at the veay base, with 3-5 pairs of sharp but short regular lateral lobes, 4-5 pairs of prominent veins;petiole slender,1.7-2.5cm.long,very deeply grooved above. Inflorescence, a generally simple several-flowered cyme, 3-5cm wide, the glabrous erect pedicels 1-2cm. long; styles 3-5, usually 4; stamens 20, large and erect; calyx hemispherical, broad and shallow, the lobes narrowly triangular or lanceolate, finely serrate, appressed pubescent, reflexed after anthesis. The fruit, borne in several-fruited clusters, or solitary, on erect or ascending pedicles 1-2 cm. long, in subglobose, or globular-pyriform, nearly 1.3 cm. long and about as thick, dark red, sometimes capped by the appressed calyx-lobes when mature; seed generally 4 or 5, about 6 cm. long, somewhat ridged on the narrow back.

Green Bay, Wisconsin, where collected by Mr. J. H. Schuette, with whose name I associate the species. While closely related to *Crataegus latisepala*, it can be seperated from it by having narrower serrate pubescent calyx-lobes, which are reflexed after anthesis, and thinner, more sharply serrate leaves, shorter pedicels, and smaller seed.

CRATAEGUS DEPRESSA. A small tree or large bush with spreading or ascending branches, and dark gray rough bark on the trunk, that on the branches being smooth and light gray; twig of the season glabrous, lustrous brown, sometimes sparingly glaucous when young, marked with numerous pale lenticels, becoming light gray the second year, armed with a few dark purple-brown thorns; buds subglobose, bright red-brown. Leaves thick, firm, glabrous, very dark green above, much paler beneath, the blades broadly ovate, or nearly orbicular on vigorous shoots, 6-9cm. long, 4-7cm. wide, rounded or subcordate at the base, abruptly acuminate at the apex, coarsely, often doubly serrate nearly to the base with acuminate gland-tipped teeth, slightly and irregularly notched; petiole slender, 2-3cm. long, finely channelled above, roughened with a few inconspicuous glands, narrowly winged at the apex by the decurrent blade. Inflorescence a few-flowered glabrous simple cyme; the bractlets few, inconspicuous and early deciduous; pedicels 1.5-2.5 cm. long, spreading in fruit; calyx cup-shaped, broad, glabrous without, pilose within at the base, the lobes short, broad, soon deciduous; styles 4-5; stamens 10, early deciduous. Fruit depressed-globose, 12-14mm. broad, 6-12mm. long, borne in clusters of 2-4 or solitary, dark red, sparingly glaucous, the cavity broad and shallow, pilose within; seed 4-5, 6-7mm. long, the lateral faces smooth, the dorsal with a deep median groove.

*Crataegus depressa* belongs to a group of which *C. haemacarpa* is the type, and most closely resembles that species, though easily separated from it by having larger and differently shaped leaves. Northern Missouri, where collected by me in 1899. I also refer here, No. 226; B. F. Bush, Dodson, Mo., July, 1899.

CRATAEGUS GRANDIS. A small tree 4-6m. in height, with dark gray scaly bark on the trunk, and long horizontal or ascending branches forming an oval or flat-topped crown; twigs glabrous, slender, those of the season light brown or russet, marked with few small inconspicuous lenticels, becoming bright gray the second year, sparingly armed with slender 4-6cm. long thorns; buds ovate, bright brown. Leaves thick and firm, dark green and shining above paler beneath, at first slightly pubescent above, especially on the midrib, soon glabrous, 5-6 pairs of prominent impressed parallel veins; the blades obovate or spathulate in outline, rounded, obtuse or adruptly acute at the apex, cuneate at the entire base, 3-5 cm. long, 2.5-3cm. wide, finely but sharply doubly serrate, seldom notched; petiole .5-2cm. long winged by the decurrent blade. Inflorescent a many-flowered glabrous corymb, sometimes minutely pubescent, especially on the calyx lobes; flowers 12-14mm. wide; calyx narrowly obconic, the entire or sparingly serrulate lobes very narrow, reflexed after anthesis; stamens uormally 20, small and slender, anthers small purplish; styles, 2-3 usually 2. The fruit, borne in compound wide-spreading clusters on spreading or pendant pedicels, is globular, 11-14mm. thick bright crimson, sometimes sparingly pruinose, marked with few inconspicuous lenticels, often capped by the persistent spreading or erect, calyx-lobes; the cavity

broad and deep; seed 2-3, more or less prominently ridged on the back, 6-8mm. long.

Illinoi : collected by me along the Wabash river, July, 1899; C. F. Johnson: Freeport; J. H. Ferriss: near Chicago.

*Crataegus grandis* is related to *C. punctata* from which it is separated by having a more glabrous inflorescence, smaller calyx, smaller and more lustrous foliage, and globular bright red fruit.

CRATAEGUS ALBEMARLENSIS. A small tree 3-7m. in height with rather smooth dark gray bark on the trunk, and horizontal or ascending branches forming an oval crown; twig of the season glabrous, chestnut-brown, marked with numerous small pale lenticels, becoming steel-gray the second year, armed with few 4-5cm.-long thorns, or unarmed. Leaves glabrous, thin but firm, dark green above, much paler beneath; the blades ovate or broadly ovate in outline, 3-5cm. long, 2.5-4cm. wide, rounded or truncate at the base, acute at the apex, finely and sharply doubly serrate nearly to the base, generally with several pairs of irregular but shallow notches, with 3-5 pairs of very slender primary veins, petioles very slender, 1.5-3cm. long, finely grooved on the upper surface. Inflorescence a nearly simple glabrous few-flowered cyme 3-4cm. wide, the very slender pedicels 1.5-2cm. long, the slender glandular bractlets very early deciduous; stamens normally 10, stout; styles 4-5; calyx broad and shallow, glabrous without, villous within at the base, the lobes narrowly triangular, entire. Fruit globose or somewhat oblong, 9-12mm. thick, dark red when ripe, capped by the erect connivent calyx-lobes; the cavity small and narrow; seed 4-5, small, 5-6mm. long.

*Crataegus Albemarlensis* is not uncommon on the moist lands bordering swamps and streams in Hyde and Pam-

lico counties, North Carolina, where it is found growing in the shade of oaks and gums or along the roadsides.

*Crataegus Holmesiana* VILLIPES. This differs from the type in the stouter, and longer pedicels which are more or less villous. The petioles are often pubescent also. Eastern Pennsylvania and New York.

CRATAEGUS FERRISSI. A large shrub or generally a small tree 4-7m. in height with ascending or spreading branches forming an oblong or oval crown, the bark on the trunk dark gray and roughened, that on the branches smooth and light gray. Twigs rather stout, glabrous, sometimes sparingly glaucous, those of the season dull brown or purplish, becoming light gray the second year, armed with rather few short 2-3cm.-long thorns. Leaves dark green above, much paler beneath, thick and firm, the blades ovate in outline, 4-6cm. long, 2-5cm. wide, acuminate at the apex, rounded, truncate or subcordate at the base, with 3 4 pairs of prominent acuminate lobes, the points of the lower pair reflexed, the sinuses acute at base, finely and acuminately glandular serrate nearly to the base; petioles slender, terete, sometimes roughened with a few small glands, 1.5-2cm. long, sometimes slightly winged at the apex, purple at base; buds ovate, brown or red-brown. Inflorescence a many-flowered corymb; calyx obconic, the narrow triangular lobes glandular serrulate; stamens 20, small; styles 4-5, usually 5. The fruit, borne in several-fruited mostly compound pendant clusters, is glabrous, crimson, oblong or pyriform, 1-1.6 cm. long, .8-1.3cm. thick, and after ripening in October persists on the trees until after the leaves have fallen; flesh thick, yellow, sweet and mealy; seed 4-5, sometimes ridged on the back, 5-7mm. long; cavity obconic, 4-5mm. broad, about as deep, the persistent calyx-tube projecting beyond the fruit, the lobes deciduous or appressed.

Northern Illinois: J. H. Ferriss; W. C. Egan.

CRATAEGUS BICOLOR. A small tree with spreading or ascending gray branches forming an oblong crown, and dark gray nearly black bark on the thorny trunk. Twigs glabrous, slender, chestnut-brown when young, armed with slender, 4-5cm.-long thorns. Leaves thin but firm, glabrous dark green above, paler beneath, the blades ovate or nearly orbicular in outline, 4-7cm. long, 3-6cm. wide, sharply but finely glandular, usually doubly serrate nearly to the base, generally with 2-3 pairs of shallow notches, acute at apex, acute or truncate at base, 3-5 pairs of principal veins; petiole 1.5-3cm. long, slender, channelled above, narrowly winged for half its length by the decurrent blade, roughened on the upper surface by several glands. Inflorescence a several-flowered compound glabrous cyme, 3-4cm. wide, the branches 1-3cm. long; calyx broad and short, glabrous, the serrate triangular lobes rising from a broad base; stamens 20; styles 4-5. The fruit, on strict or spreading .8-1.8cm.-long pedicels, disposed singly or generally simple clusters of 2-4, is subglobose, generally slightly oblong 11-13mm. long, green or greenish mottled with red; calyx lobes generally sessile on the mature fruit, their base only persistent; flesh firm, white; cavity broad, very shallow, obconic; seed 4-5, small, 5-6mm. long, deeply grooved on the back, apical in the fruit.

*Crataegus bicolor* is found abundantly in sunny oak woods, especially on well-drained southern slopes, in the vicinity of Micaville and Spruce Pine, Mitchell County, North Carolina. It is frequently associated with *C. cruenta* which it resembles in habit and foliage, and from which it is seperated by having more numerouss seed and greenish fruit.

CRATAEGUS ATRO-PURPUREA. A small tree 4-7m. in height with dark gray or blackish scaly bark on the

trunk, and long spreading or ascending branches forming an oval crown. Twig of the season glabrous, chestnut-brown, becoming gray the second year, sparingly armed with 4-5cm-long thorns. Leaves glabrous, thin but firm, dark green and shining above, somewhat paler beneath, blades ovate or deltoid in outline, or on vigorous shoots nearly orbicular, 4-7cm. long, 3-7cm. wide, acute or obtuse at the apex, obtuse or truncate at the base, sharply doubly serrate, the teeth gland-tipped, sometimes with 2-4 pairs of very shallow notches, usually 4 pairs of primary veins; petiole slender, 2 3cm. long, grooved on the upper surface, winged at the apex by the decurrent blade. Inflorescence cymose, glabrous, sparingly glaucous, few-flowered, 3-5cm. wide, bractlets few, early deciduous; pedicels ascending, 1-2cm. long; calyx glabrous, short and broad, the lobes narrowly triangular; stamens 20, small; styles 3 exceptionally 4. The fruit, ripening late in October or in November after the leaves have fallen, and long persistent, in clusters of 2-5 on strict .7-2cm'-long pedicels, is oblong, 1-1.5cm.-long, 8-11mm. thick, rounded at the ends, dark red or purplish, scantily glaucous, capped by the nearly sessile persistent reflexed calyx-lobes; cavity obconic, broad and shallow; seed generally 3, somewhat apical in the fruit, 6-8mm. long, white, with shallow grooves on the rounded back, somewhat attenuate at the base; flesh firm, white, sweet. The foliage turns a dull yellow or brown and falls soon after the first severe frost.

*Crataegus atro-purpurea* frequents fields, roadsides and dry sunny woods in Yancy county, North Carolina, at an elevation of about 800m., but is not common.

It belongs to a group of which C. macrosperma and C. haemacarpa can be regarded as the types. From the former it is seperated by the different outline of the

foliage, smaller fruit and fewer and smaller seeds; from the latter by having an oblong fruit with white flesh and fewer seed.

CRATAEGUS CARNOSA. A small bushy tree or a shrub with numerous ascending branches from near the base. Twigs glabrous, chestnut-brown, rather thick, straight or somewhat flexuous, armed with very numerous 4-6cm.-long thorns. Leaves glabrous, membranaceous, the blades ovate or elliptic in outline, 4-6cm. long, 2-4cm. wide, obtuse or acuminate at the apex, obtuse or acute at base, sharply serrate, 3-5 pairs of lobes from near the base, with 4-5 pairs of ascending primary veins; petiole 1.5-3cm. long, grooved on upper surface, narrowly winged for half its length or more by the decurrent blade. Inflorescence a glabrous several-flowered compound cyme, 3-4cm. wide; pedicels very short .8-1.5cm. long, erect; stamens 20, small; styles 2-3; calyx small, glabrous, the triangular lobes usually entire. Fruiting pedicels solitary or in clusturs, sometimes compound, of 2-5, spreading or ascending, .8-1.4cm. long. Fruit globose-pyriform, flattened at the apex, concave at base, generally broader than long, 1.2-2cm. broad, 1-1.8cm long, bright orange-red when ripe in October and persistent for sometime after the fall of the foliage (in 1899 persistent until late in December); flesh orange-yellow, thick and juicy; the sessile, reflexed calyx-lobes often persistent; cavity obconic, shallow; seed 2, exceptionally 3 or 4, deeply gooved on the back, 9-7mm. long. The leaves turn a dull yellow or brown and fall early, after the first severe frost.

I have seen *Crataegus carnosa* only in Yancey county, North Carolina, where it grows on roadsides and in sunny upland woods, often associated with *C. atro-purpurea* but is by no means common. It is easily seperated from

the other 20-stamened species of this region by its very narrow, sharply lobed leaves.

CRATAEGUS PRISMATICA. A small tree 3-5m. in height, or sometimes a large shrub, with short spreading branches forming a small oval or flattened crown, and dark, nearly black scaly bark on the trunk. Twigs of the season slender, flexuous or nearly straight, red-brown, becoming gray the second year, glabrous, armed with few 4-5cm.-long thorns. The glabrous, rather pale green leaves are thin but firm in texture, the blades elliptic, ovate, obovate or even spathulate in outline, 3-4.5 cm. long, 1.5-4cm. wide, rounded or obtuse at apex, rounded or cuneate at the serrate base, sharply glandular serrate, doubly serrate above the middle or with 1-3 pairs of shallow notches, 3-4 pairs of impressed primary veins; petiole 1.5-2cm. long, slender, channelled above, winged at the apex by the decurrent blade, roughened by 2-3 pairs of stalked glands; winter-buds globose, bright red-brown. The inflorescence is a nearly simple, glabrous few-flowered cyme, the short pedicels bracteate with 2-3 lanceolate glandular-margined bracts; calyx glabrous, the tube short and broad; stamens small, generally 10; styles 3, seldom 4; fruiting clusters 1-4-fruited, the bright red erect pedicels 1-1.5cm long, about the length of the fruit. Fruit globular-pyriform, somewhat thicker than long, 12-15mm. thick, 11-15mm. long, 3-angled in cross section, bright red; calyx-lobes lanceolate, sharply glandular-serrate above the middle, on the mature fruit slightly stalked, reflexed, bright red; the broad shallow cavity obconic; flesh firm, yellow, mealy; seed usually 3, seldom 4, apical in the fruit, about 6mm. long, 3-4mm. thick laterally, deeply dorsally groovod. The fruit falls with or soon after the fall of the foliage, late in October or the first half of November. The

foliage becomes bright red, mottled with orange and yellow early in October and is tardily deciduous after the first heavy frost.

*Crataegus prismatica* resemble in habit and foliage *C. flavo-carnis* from which it can be seperated by having smaller and more obtuse leaves, pyriform fruit, fewer styles and seed. Banks of streams Madison county, North Carolina, especially along the French Broad river.

CRATAEGUS CRUENTA. A slender tree 5-7m. high with short ascending or spreading branches forming an oval or oblong crown, and dark gray or blackish rough bark on the trunk. Twigs of the season glabrous, slender, light brown, becoming steel-gray the second year, armed with very slender 2-4cm.-long chestnut-brown or purplish thorns. Leaves thin, bright green, glabrous, the blades broadly ovate, deltoid or nearly round in outline, 3-5cm. long, 2.6-5cm. wide, obtuse rounded or truncate at the entire base, acute at the apex, sharply and finely doubly serrate, with 2-3 pairs of shallow notches, 3-4 pairs of primary veins; petiole slender, 1.5-2.5cm. long, nearly terete, on vigorous shoots roughened with glands. Inflorescence a few-flowered compound cyme 5-6cm. wide, glabrous, somewhat glaucous; calyx glabrous, the tube broad and short, the entire triangular lobes rising from a broad base. reflexed after anthesis; stamens 20, small; styles generally 3. The fruit, solitary or in clusters of 2-4, on strict slender pedicels .8-1.8cm. long, is globose or somewhat attenuate at base, 9-13mm. long, about as thick, bright red, sparingly glaucous, capped by the short-stalked generally persistent reflexed calyx-lobes; flesh yellowish white, rather thin, firm, sweet; seed usually 3, generally slightly grooved on the rounded back, 6-7mm. long. The foliage, which is long persistent ac-

quires after frost many shades of yellow, red and purple and falls before the fruit which often persists until late in winter.

*Crataegus cruenta* is frequent in Yancey county, North Carolina, between Spruce Pine and Micaville, growing in sunny upland woods beneath oaks and hickories. It can be separated from the other 20-stamened forms of this region with which it is associated, by its smaller and broader leaves, and the small, bright red, usually 3-seeded fruit.

CRATAEGUS TRISPERMA. A tree 5-8m. in height with ascending or spreading branches forming an oblong crown, and a slender trunk with dark gray nearly black bark, armed with few thorns. Twig of season slender, glabrous, bright brown, becoming dull gray the second year, marked with pale lenticels, straight or nearly so, armed with few slender 4-5cm.-long thorns. Leaves glabrous, thin, bright green above, somewhat paler beneath, the blades ovate or broadly ovate in outline, 4-7cm. long, 3.5-6cm. wide obtuse or truncate at the base, acute or obtuse at the apex, sharply and coarsely doubly serrate, with 3-4 pairs of shallow lobes; petiole 1.5-2.5cm. long, grooved above, margined at the apex by the decurrent blade, roughened with several pairs of red glands, purplish at base. The inflorescence is a glabrous, generally simple cyme, 3-5cm. wide; pedicels .6-2cm. long; calyx obconic, the short broadly triangular lobes entire; styles 3-4, generally 3; stamens small, 10. The fruit, on erect or spreading .6-2cm. long pedicels, usually in clusters of 3-5, is globose, 10-13mm. thick, bright yellow or orange when ripe in October, capped by the short, sessile calyx-lobes; flesh rather thick and firm, white or pale yellow; seed generally 3, nearly central in the fruit, 5-7mm. long, deeply grooved on the rounded back. The leaves turn a

dull yellow or brown and fall soon after frost, before the fruit which is largely persistent until December.

The species above proposed is abundant in Yancey County, North Carolina, between the North and South forks of Toe river, at an elevation of about 800m, growing on roadsides and on sunny wooded slopes. It is separated from the other 10-stamened species with which it is associated by its globular fruit. It is most closely related to *C. riparia* of the Little Tennessee river basin, which differs from it in having larger fruit, more numerous nutlets, and a differently shaped leaf.

CRATAEGUS CRASSA. A small tree 4-6m. in height with slender bright red-brown twigs armed with numerous slender 3-4cm.-long thorns. Leaves dark green, thick and firm, pubescent above along the midrib, pubescent beneath when young, at length nearly glabrous, except for small tufts of hairs in the axils of the primary veins, obovate or nearly orbicular, obtuse or rounded at the apex, rounded or abruptly contracted at the entire base, glandular crenate or serrate, sometimes doubly so, with 1-3 pairs of shallow obtuse notches above the middle, 2-3 pairs of prominent ascending deeply impressed primary veins; petiole 1-2cm. long, slender, nearly terete pubescent above, broadly winged at the apex. Inflorescence a compound many-flowered cyme 4-5cm. wide; flowers about 1.5cm. wide, on slender erect glabrate pedicels; calyx small, about 3mm. long, pubescent, as well as the narrowly triangular acute glandular serrate lobes; styles 4-5; stamens 20, persistent; fruit oblong, 9-12mm. long, dark red, capped by the nearly sessile persistent calyx-lobes; flesh thin, sweet; seed usually 5, 4-5mm. long, smooth on the back and sides; cavity nearly 5mm. wide.

Eastern Pennsylvania. *C. crassa* belongs to the flava group. It is separated from *C. flava*, however, by its smaller fruit, which is not pyriform, and its more numer-

ous flowers.

*Crataegus deltoides.* A straggling shrub, or occasionally a small tree, with spreading sparingly armed branches, and slender glossy brown twigs which do not become gray until the third year. Leaves thick, firm, glabrous, dark green above, much paler beneath, deltoid or nearly orbicular in outline, 3-5cm. in diameter, truncate or subcordate at the base, sharply doubly glandular serrate, with severl pairs of shallow notches, 3-5 pairs of prominent divergent lateral veins: petiole 1.5-2cm. long, nearly terete, glaucous. Inflorescence a 3-6-flowered nearly simple cyme, 3-5cm. wide; pedicels stout, erect, glabrous, glaucous; flowers 15-17mm. wide; calyx cup-shaped, very broad, the very prominent oblong or ligulate lobes glabrous, serrate or entire, erect or spreading soon after anthesis; stamens normally 10; styles 3-4; fruit large, 15-17mm. thick, globular or somewnat depressed, concave at the base, dark red, sparingly pruinose; seed 3-4, ridged on the back, the lateral faces plane; cavity 6-7mm. wide.

*Crataegus deltoides* is frequent in southeastern Pennsylvania, and the adjoining parts of Maryland. It is related to *C. pruinosa* from which it is separated by its larger fruit and foliage, and fewer stamens.

CRATAEGUS PORRACEA. A tree 5-7m. high with spreading or ascending branches. forming an oval or rounded crown. Twig of the season glabrous, red-brown, becomig gray the second yeay, armed with short 2-3cm.-long thorns. Inflorescence a small, 3-4mm.-wide glabrous corymb, 6-10-flowered; flowers 15-17mm. wide; calyx obconic, the lobes narrowly triangular, entire or nearly so, reflexed after anthesis, generally deciduous before autumn: stamens 10; styles 3-5; fruit oblong, 10-13mm. long, dark red, sparingly glaucous; seed 3-5, furrowed on the back, the lateral faces plane; fruiting pedi-ascending or spreading .8-15cm. long; fruit persistent

after the leaves have fallen. Leaves thin, glabrous, bright green above, somewhat paler beneath, the blades small, 3-4.5cm. long, ovate, acute at apex, rounded or truncate at base, very finely and sharply doubly serrate all around, 4-5 pairs of sharp lateral lobes, the tips of the lower usually reflexed, 4-6 pairs of straight parallel veins; petiole slender, nearly terete, 1-1.8cm. long.

The species above proposed is related to *C. tenuifolia* from which it is separated by the smaller foliage and smaller fruit and corymbs. Northeastern Pennsylvania and the adjacent part of New York. The type material was collected by C. F. Saunders, near Susqnehanna, Pa.

CRATAEGUS ALBICANS. A large shrub or a small tree, 4-7m. in height with spreading or ascending branches. Twig of the season glabrous, bright red-brown, sparingly glaucous, at length dull gray, armed with few 3-5cm.-long thorns. Leaves glabrous below, rough above, at least when young, with scattered very short hairs, especially along the midrib, firm in texture, the blades ovate in outline, 5-8cm. long, rounded or truncate at base, acute at apex, sharply and finely doubly serrate, with 4-6 pairs of shallow acute lobes; petiole slender, terete or nearly so, 2-3cm. long. Inflorescence a nearly simple cyme, 5-6cm. wide, 5-10;flowered, glabrous except the calyx-lobes; flowers about 17mm. wide; stamens normally 10; styles 4-5; fruit globular, 1-1.3cm. thick, dark red, glaucous, capped by the nearly sessile, spreading calyx-lobes, which are broadly triangular, usually entire, and pubescent on the upper surface; seed 4-5, 5-6mm. long, grooved on the back, the lateral faces plane; flesh thick, firm, yellowish, very sweet. The tips of the calyx-lobes often break off and the edges of the persistent base become involute,

*Crataegus albicans* has been collected in eastern Michigan by Mr. O. A. Farwell. It is related to *C. pruinosa*. *Issued July 1, 1901. N. C. Nat. Hist. Survey.*

# A PECULIAR IRON OF SUPPOSED METEORIC ORIGIN, FROM DAVIDSON COUNTY, NORTH CAROLINA.

JOSEPH HYDE PRATT.

The iron to be described was found six or seven years ago by Mr. W. R. Harris on a hill-side that rises just east of the Lexington-Troy road about half a mile south of Cid P. O., Davidson County, N. C.

The iron originally weighed 13 lbs. 14 ozs., and its shape was somewhat oblong, 9 inches long by 6½ inches in its widest part, and 2 inches thick in the center, but becoming much thinner than this around its outer edge. Surrounding the iron is a more or less oxidized surface which in some places extends into the iron for a quarter to nearly half an inch. Its surface is more or less pitted and it has very evidently lain in the earth for some time.

The iron has been cut and polished and the surfaces tested by etching but they showed neither the Widmannstatten figures nor the Neumann lines. Mr. Tassin of the United States National Nuseum who has also experimented with this iron states that "the etched surface peresents a granular or stippled appearance overlaid with a network of fine lines. These lines are apparently without orientation and are so nnmerous as to give the iron a decidely fibrous appearance." He also states that the "fractured surface shows traces of what is apparently an octahedral cleavage." This etched surface is dissimilar to the other so-called structureless meteorites that have been tested in the National Museum except the Scriba iron* which was found at Scriba, Oswego County, New York, in 1895, whose etched surface is very similar. This dissimilarity is of course against the meteoric nature of this iron as these tests are the unquestionable ones for a meteoric iron, but at the same time some irons that are known meteorites have not given them.

*Am. J. Sci. Vol. XL, 1841, p. 366.

In comparing the etched surface of this iron with those of a series of wood burned, coke and bloomery irons, Mr. Tassin, who has made a great many etching tests on irons of this sort, states that it is radically different in appearance from any of these manufactured irons that he has tested. Of course it must be taken into consideration that these manufactured irons that were tested were for the most part manufactured by established methods and for special purposes. What the effect of etching would be on a polished surface of some of the old charcoal irons which have been obtained by crude processes is not known. That a mass of cast iron of this weight and shape should have been carried and left in this section which is 15 to 29 miles from any railroad and 50 or a 100 miles from any known furnace is rather improbable.

This iron has been examined by a number men who are thoroughly familiar with all kinds of manufactured irons and they state that it is entirely different from any iron that they have ever seen. Mr. Kerr, a foundryman of Durham, North Carolina, who cut off a piece of this iron, stated that it was unlike any that he had ever seen and did not cut like any iron known to him. It is tough and a piece can only be chipped off of the main mass with difficulty; and it is not brittle like cast iron, but on the contrary is somewhat malleable.

There is considerable free carbon in the iron which is very apparent on a fresh fracture, and it leaves a black stain on any thing touching it. This interferes to some extent with the etching and partly explains the stippled appearance of the etched surface.

In its chemical composition it is different from the majority of the meteoric irons in having a small amount of nickel and containing some free carbon. Material was obtained for analysis from the interior of the iron where it showed no alteration, the turnings from this portion of the iron having been carefully selected at the time the iron was cut It has been analyzed by Dr. Baskerville,* who used the method of analysis described below.

*University of North Carolina.

METHOD OF ANALYSIS.

A weighed portion of the iron turnings were treated with aqua regia, and brought to a syrup by direct heat, the nitric acid being driven off by addition of concentrated hydrochloric acid and evaporating several times to dryness on the water bath. The silica was finally rendered insoluble by heating at 110°C until all hydrochloric acid fumes were removed. It was then treated with dilute hydrochloric acid and boiling water and filtered. The filtrate was made up to 500 c. c. and reserved for furthur use. The precipitate after thorough washing was burned in a platinum crucible. After decarbonization it was fused with five parts sodium carbonate and one part potassium nitrate, taken up in dilute hydrochloric acid, evaporated to dryness and heated at 110° C, as above to render silica insoluble. The residue was then treated with hydrochloric acid and hot water, filtered and washed. The filtrate was made up to 250 c. c.

The precipitate was burned and the silica driven off by hydrofluoric acid. The residue was tested for titanium according to method of Weller* as modified by Hillebrand† and Dunnington,‡ after fusion with potassium disulphate, but none was detected.

The filtrates were tested for sulphur and phosphorus according to the ordinary methods of testing for the oxidized compounds of those elements, but not a trace of either was observed.

An aliquot part of the first filtrate (500 c. c.) was precipitated with ammonium hydroxide, filtered, and the precipitate washed, burned and weighed as $Fe_2O_3$. The iron in another portion was precipitated as before, and then re-dissolved in dilute sulphuric acid, passed through a granulated zinc column reductor and titrated with standard potassium permanganate solution This method gave the same results as were obtained by the gravimetric, thus showing the absence of alum-

*Ber. D. Chem. Ges. 15,2592

†J. Am. Chem. Soc. 17, 718.

‡J. Anal. and Appl. Chem. 5, 39.

inum. As the ammoniacal filtrate possessed no blue color, copper was reported absent.

The second filtrate (250 c. c.) was similarly treated and the iron combined with carbon and silicon determined which amounted to 1.81 per cent, this with the metallic iron gave the total iron present. After calculating the silicon (0.62 per cent) as iron silicide $FeSi_2$, that amount of iron (0.62 per cent) was taken from 1.81 per cent and the difference (1.17 per cent) was regarded as iron combined with carbon. This was subsequently checked as noted below.

Manganese was determined from an aliquot part of the two filtrates by the basic acetate method and long boiling of filtrate with an excess of bromine. The precipitate was burned and weighed as $Mn_3O_4$.

Nickel and cobalt were separated from the filtrates which had been thoroughly oxidized and changed to chlorides by repeatedly extracting the ferric chloride by means of pure ether in a Rothe's pipette*. The Nickel and cobalt chlorides were then separated according to Blair† viz: by precipitation of the cobalt as the potassium cobaltinitrite, $K_6Co_2(NO_2)_{12}$.

For total carbon two grams were dissolved in the double potassium and copper chloride. The undissolved graphite and carbide were caught on asbestos in a platinum Gooch crucible, dried and weighed.‡ The total carbon was determined by combustion in the Shimer§ apparatus.

The difference in the weight of the carbon found and the total undissolved residue was 1.17 per cent, which is the same as the combined iron not reported as combined with the silicon. (see above).

Every effort was made to determine the combined carbon, but the material was not attacked by nitric acid, even after digestion at 100° C. for twelve hours. The colorimetric method failed entirely.

*Mitt. a. den Konig. Teh. Vers. zu Berlin, 1892, part iii-Blair, 3rd. edit. p. 315.

†Ibid. p. 185.

‡Blair. ibid. p. 162.

§J. Am. Chem. Soc. 21, p. 557.

No effort was made to determine occluded gases.

The results of the analysis are given below:

| | | |
|---|---|---|
| Iron | 93.89 | Iron, metallic ..... 92.10 |
| | | Iron combined with carbon 1.17 |
| | | Iron combined with silicon .62 |
| | | Manganese ..... .92 |
| | | Nickel ..... .30 |
| | | Cobalt ..... .34 |
| | | Sulphur ..... None |
| | | Phosphorus ..... None |
| | | Aluminum ..... None |
| | | Titanium ..... None |
| | | Silicon ..... .62 |
| | | Carbon ..... 3.88 |
| | | Copper ..... None |
| | | 99.95 |

The above results show the iron to be considerably different in its chemical composition from the manufactured irons and, while the percentage of nickel is low, that does not preclude it from the meteorites. The presence of the nickel even in this small amount and of the cobalt with the total absence of sulphur, phosphorus, titanium and aluminum are strong factors against its being a manufactured iron and for its meteoric origin.

The Scriba iron already refered to, which is similar in structure to this iron, is decidedly different in its chemical composition. An analysis of this by Prof. Charles U. Shepard in 1841 gives the following results:

| | |
|---|---|
| Iron | 99.68 |
| Silicon | 0.20 |
| Aluminum | Trace. |
| Total | 99.88 |

This analysis, except as showing the iron to be a very pure one, does not add any weight to its meteoric character but the rather isolated country in which it was

found and its peculiar configuratoin have led to the belief that it is of meteoric origin although this is questioned by many.

The analysis, however, of the iron from Davidson County supports its meteoric origin and then again, its structure, while being different from that of the other known meteorites, together with the more or less isolated region in which it was found makes it very probable that this iron is of meteoric origin.

From the locality where this iron was found, namely, in the vicinity of Cid P. O., Davidson County, North Carolina, the name proposed for it is the "Cid" iron.

*N. C. Geological Survey.*

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## A METHOD OF LAYING OUT GRADES FOR SEWERS BY AID OF THE TRANSIT.

WILLIAM CAIN.

The transit method, as I shall term it, of laying out sewer grades, presupposes the use of the level in determining the elevations, so that both transit and level are used as in the ordinary method. The peculiarity of this method consists in this, that ultimately the transit telescope is pointed along a line parallel to the grade of the invert of the sewer and from this line of sight, distances can be measured down to locate any point desired as explained below.

As this is a very practical subject, the successive steps to follow will be given by aid of a particular example.

In the figure, suppose the elevation of the grade of the invert at the Man Hole, Station 0, to be 476.90 and that 420 feet from Station 0, or at Station 4+20, the elevation of grade is 488.79, giving a rise of 2.83 feet in a hundred. This much

is obtained from the profile. Suppose the trench partially dug, the depth of cutting being approximately given from the profile and that at intervals, say of 40 (more or less), *boards* are placed across the ditch and firmly embedded in the earth. These boards may be 6 or 8 inches in depth and their sides must be as near vertical as the eye can place them.

The successive steps are then as follows:

(1) Place the transit on the center line of the trench a certain convenient distance back of Station 0, as 10 feet or some multiple of 10. The elevation of the grade of invert just under the transit, supposed in the figure 10 feet back of Station 0, is thus, 476.90—0.28=476.62.

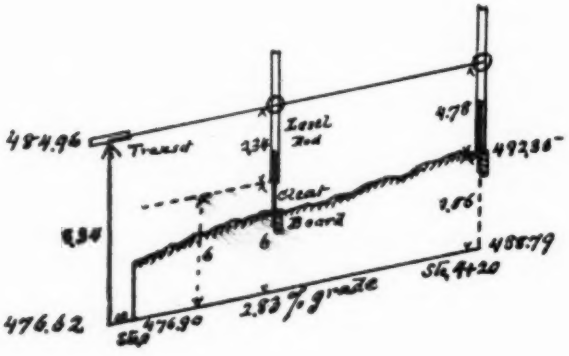
(2) With the transit sighted along the center line of trench set a rod or pencil on an edge of each "board," in line; mark the point with a pencil and then nail on a *cleat* to the board with a vertical edge on the point. After one nail is driven in cleat by aid of the vertical thread of the transit telescope, the top of the cleat is moved over until the edge is truly vertical. These cleats are say, 4 in. by ⅜ in. by 4 feet, though often longer cleats are needed.

If house connections are to be put in, the distance from board to board is measured with a steel tape and recorded, as the connections are generally located by the constructor with reference to the boards, numbered 1, 2, 3, . . . .

(3) The level is now set up, about midway of the length of trench considered and the elevation of the board at station 4+20 is found, also the elevation of the telescope of the transit instrument. To find the last by the level, place the foot of the level rod on the ground and hold the rod against the transit axis and take the reading of the rod by the level and then compute the elevation of the foot of the rod. On adding to this, the height to axis of transit telescope as measured on the rod, we find the elevation of the transit axis (484.96 in figure). If the transit telescope has a level on it, it is best to bring the bubble to center, sight at a level rod held on a convenient bench mark and add the reading to the elevation of the bench to get the elevation of the transit axis.

(4) The difference between the elevations of the transit axis and that of the grade under it, is thus, 484.96—476.62= 8.34. The difference between the elevations of board and grade at station 4+20 is, 492.35—488.79=3.56. Therefore, set a target on the level rod at, 8.34—3.56=4.78, hold on board at 4+20 and bisect target with horizontal hair of transit telescope. The latter is now evidently sighted along a line parallel to grade, since the line of sight is 8.34 ft. above grade at transit and it is 3.56+4.78=8.34 ft. above grade at station 4+20.

Tack an envelope on the cleat at station 4+20, with its top on the line of sight, for reference, and before each subsequent




observation, test the line of sight and correct if need be, by its aid. If the cleat is not long enough, nail on a strip to the board sufficiently long and tack the envelope on that.

(5) Suppose it desirable to drive nails in a certain number of cleats, 6 feet above grade; set the target on the level rod at 8.34—6=2.34. Then let a common hand hold a hammer on the vertical edge of a cleat that is in the vertical plane containing the center line of the trench and rest the foot of the rod on it. The rodman then balances his rod carefully on the hammer head and moves it up or down, the hammer following,

until the transit man bisects the target. A pencil mark is then made across the hammer head and a nail driven horizontally into the edge of the cleat. The rod is held on it at once, and its position is altered if the target is not bisected. Points 6 feet above grade are similarly determined at other boards and eventually the constructor stretches a stout cord from nail to nail, to give him a continuous line, 6ft. above grade, to work from. Similarly for 7 ft., 8 ft..,lines, two nails being put in on a cleat where there is a change from one cord parallel to grade to another at a different distance from the invert.

(6) The line is now sufficiently checked by taking the elevations of the nails at station 0 and 4+20, by the level (which has not been moved). If these elevations are the assumed heights (6 ft., etc.) above the grades (476.90 and 488.79) at those stations, the work is correct.

The elevations have been given to hundredths in the above. They may be carried to thousands if preferred. It would seem that working to hundredths is sufficiently close unless the grade is very light. On steep grades errors of several hundredths in fact may be allowed, if there is nevertheless, a continuous fall in the sewer.

Man Holes are placed every few hundred feet, so that in case of obstruction, from any cause, including breaking of the sewer or pipe, the part of the line obstructed can be quickly located by sighting through the pipe. The alignment both vertically and horizontally should be carefully done to afford a free sight. The method of locating the nails to string the cords on, outlined above is rapid in its working and does not take half the time required by the common method, which consists in getting by aid of the level, the elevations of each board, computing grades there, elevations of cords and thus heights from board to cord. These distances are then laid off on the cleats, which are then nailed on by aid of the transit, "in line." The rod reading on any nail should then be, height of instrument minus elevation of nail. Nine columns for computation are required in the field book and

the work is tedious, but it is systematic and little liable to mistake. In many cases, this method has to be used, especially in crossing a ridge. Where only two or three cleats have to be put in, this old method takes no more time than the transit method, but where a dozen cleats have to be set and the nails placed in them, the transit method, including all the preliminary work common to both methods, can be done in less than half the time.

By the latter method, it is seen from the profile, whether it is best to set the transit at the upper or lower end of the line, or perhaps the state of the filling in a completed trench, may determine the matter. Although station 0 was assumed above to be at a Man Hole, the reasoning and method is the same if it is anywhere on the line.

A possible source of error suggests itself by this method: does the transit telescope, as the object glass moves out or in, have its successive lines of sight (along the horizontal hair,) to lie in the same plane? This is doubted by some as the object glass is supposed to be placed mainly with reference to the vertical hair and all error of eccentricity of object glass, is said, by some makers*, of the transit, to be thrown into the horizontal hair.

The following test was made for the Fauth & Co. transit used in laying sewer grades at Chapel Hill, N. C.

Using 100 feet stations, pegs were put in at stations 0, 0+50. 1, 2, 3, 4, 5, 6 and their elevations found by running a line of levels from station 0 to 6, setting up the level, in turn, midway between two adjacent pegs and sighting on them, the true level was found as usual, so that the curvature of the earth was considered.

These elevations are given in column 2 of the following table.

*See *Engineering News* for May 23rd., 1901, page 377.

| Station | Elevations by level | Elevations by transit | Difference |
|---|---|---|---|
| 0 | 10.000 | 10.000 | 0.000 |
| +50 | 9.204 | 9.205 | +0.001 |
| 1 | 8.742 | 8.739 | −0.003 |
| 2 | 7.960 | 7.961 | +0.001 |
| 3 | 6.220 | 6.225 | +0.005 |
| 4 | 6.125 | 6.121 | −0.004 |
| 5 | 7.809 | 7.808 | −0.001 |
| 6 | 9.712 | 9.712 | 0.000 |

Station 6 is thus found to be 0.288 ft. below station 0.

The *transit* was now set up at station 0 and the height of telescope axis above the peg at 0, found to be 4.955. The correction for curvature and refraction of earth being 0.007 for six hundred feet, a target was now set on the rod at, 0.288+4.955+0.007=5.250 and the rod held at station 6.

The transit telescope was made next to bisect this fixed target, so that it was now sighted along a true horizontal line at station 0. The object glass alone was moved to get the rod readings on stations 5,4,3,...., in turn; but just before taking a sight, the fixed target at station 0 was sighted at, to correct any possible movement of the telescope. The correction for curvature was now subtracted from each reading in turn to get the elevations in column 3. The amount and character of the small errors in column 4, point rather to personal and instrumental errors than to any eccentricity of the object glass.

It is plain that this transit can be safely used for ordinary levelling or in establishing grades by "the transit method". Any transit can be similary tested.

I have heard this transit method spoken of as a possibility, thirty yeare ago for railroad grades, but never knew it put in practice. It is very expeditious, taking only half the time of the level method.

# THE CONE OF THE NORMALS AND AN ALLIED CONE FOR CENTRAL SURFACES OF THE SECOND DEGREE.*

ARCHIBALD HENDERSON, M.A.

1. The cone of the normals, which we shall designate throughout by C, will first be deduced for the ellipsoid given in its simplest form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The well known equation of the normal to the surface (1) at a point $(f, g, h)$ is given by

$$\frac{x-f}{\frac{f}{a^2}}=\frac{y-g}{\frac{g}{b^2}}=\frac{z-h}{\frac{h}{c^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

2. From a point P, $(x_0, y_0, z_0)$, in space, six normals may in general be drawn to the ellipsoid.

If the normal

$$\frac{x-f}{\frac{f}{a^2}}=\frac{y-g}{\frac{g}{b^2}}=\frac{z-h}{\frac{h}{c^2}}=\lambda'$$

pass through the point P $(x_0, y_0, z_0)$, we have

$$\frac{x_0-f}{\frac{f}{a^2}}=\frac{y_0-g}{\frac{g}{b^2}}=\frac{z_0-h}{\frac{h}{c^2}}=\lambda \quad \ldots\ldots\ldots\ldots(1)$$

Then $x_0-f=\frac{f\lambda}{a^2}$, $\therefore f(a^2+\lambda)=a^2x_0$, $\therefore f=\frac{a^2x_0}{a^2+\lambda}$.

*Dissertation presented to the Faculty of the University of North Carolina for the degree of Doctor of Philosophy, May 1901.

Similarly $g=\dfrac{b^2y_0}{b^2+\lambda}$, $h=\dfrac{c^2z_0}{c^2+\lambda}$. Since the point $(f, g, h)$ is on the ellipsoid

$$\frac{a^2x_0^2}{(a^2+\lambda)^2}+\frac{b^2y_0^2}{(b^2+\lambda)^2}+\frac{c^2z_0^2}{(c^2+\lambda)^2}=1.$$

This equation is of the sixth degree in $\lambda$ and, since to every value of $\lambda$ corresponds a particular normal, the number of normals from a given point to the ellipsoid is, in general, six.

3. The six normals from a point P $(x_0, y_0, z_0)$ to the ellipsoid lie on a cone of the second degree, with vertex at P.

The equations of the normal to the ellipsoid at the point $(f, g, h)$ are

$$\frac{x-f}{\frac{f}{a^2}}=\frac{y-g}{\frac{g}{b^2}}=\frac{z-h}{\frac{h}{c^2}}=\lambda$$

Since this normal passes through the point P $(x_0, y_0, z_0)$ we have

$$\frac{x_0-f}{\frac{f}{a^2}}=\frac{y_0-g}{\frac{g}{b^2}}=\frac{z_0-h}{\frac{h}{c^2}}=\rho.$$

Next let us eliminate $(f, g, h)$. By subtraction we obtain $\dfrac{x-x_0}{\frac{f}{a^2}}=\lambda-\rho$. Also since $\dfrac{x_0-f}{\frac{f}{a^2}}=\rho$, it follows that $f=\dfrac{a^2x_0}{a^2+\rho}$;

hence $x-x_0=\dfrac{x_0}{a^2+\rho}(\lambda-\rho)$. Then we have $a^2+\rho=\dfrac{x_0}{x-x_0}(\lambda-\rho)$;

*i. e.*, $$\left(1+\frac{x_0}{x-x_0}\right)\rho-\left(\frac{x_0}{x-x_0}\right)\lambda+a^2=0.$$

Similarly, $\left(1+\frac{y_0}{y-y_0}\right)\rho-\left(\frac{y_0}{y-y_0}\right)\lambda+b^2=0$. And

$$\left(1+\frac{z_0}{z-z_0}\right)\rho-\left(\frac{z_0}{z-z_0}\right)\lambda+c^2=0.$$

Eliminating $\rho$ and $\lambda$ between these last three equations, we have

$$\begin{vmatrix} 1+\frac{x_0}{-x_0}, & \frac{x_0}{x-x_0}, & a^2 \\ 1+\frac{y_0}{y-y_0}, & \frac{y_0}{y-y_0}, & b^2 \\ 1+\frac{z_0}{z-z_0}, & \frac{z_0}{z-z_0}, & c^2 \end{vmatrix} = 0.$$

Subtracting the second column from the first, we obtain

$$\begin{vmatrix} 1, & \frac{x_0}{x-x_0}, & a^2 \\ 1, & \frac{y_0}{y-y_0}, & b^2 \\ 1, & \frac{z_0}{z-z_0}, & c^2 \end{vmatrix} = 0. \quad - - - - - \quad (1)$$

Multiplying the first row by $\frac{x-x_0}{a^2}$, the second row by $\frac{y-y_0}{b^2}$, the third row by $\frac{z-z_0}{c^2}$, we have

$$\begin{vmatrix} \frac{x-x_0}{a^2}, & \frac{x_0}{a^2}, & x-x_0 \\ \frac{y-y_0}{b^2}, & \frac{y_0}{b^2}, & y-y_0 \\ \frac{z-z_0}{c^2}, & \frac{z_0}{c^2}, & z-z^0 \end{vmatrix} = 0. \quad - \quad - \quad - \quad (2)$$

Adding the second column to the first and interchanging columns two and three, we obtain a third form:

$$\begin{vmatrix} \frac{x}{a^2}, & x-x_0, & \frac{x_0}{a^2} \\ \frac{y}{b^2}, & y-y_0, & \frac{y_0}{b^2} \\ \frac{z}{c^2}, & z-z_0, & \frac{z_0}{c^2} \end{vmatrix} = 0. \quad - \quad - \quad - \quad (3)$$

Transforming to parallel axes through the point $P(x_0, y_0, z_0)$ as origin by the substitution $x-x_0=x$, $y-y_0=y$, $z-z_0=z$ and employing form (1) above we obtain

$$\begin{vmatrix} 1, & \frac{x_0}{x}, & a^2 \\ 1, & \frac{y_0}{y}, & b^2 \\ 1, & \frac{z_0}{z}, & c^2 \end{vmatrix} = 0. \quad - \quad - \quad - \quad (4)$$

$$\text{or} \qquad (b^2-c^2)\frac{x_0}{x} + (c^2-a^2)\frac{y_0}{y} + (a^2-b^2)\frac{z_0}{z} = 0. \quad - \quad (5)$$

$$\text{or} \quad x_0(b^2-c^2)yz + y_0(c^2-a^2)zx + z_0(a^2-b^2)xy = 0. \quad - \quad (6)$$

which is the equation of a cone of the second degree, having its vertex at the new origin, P.

4. The cone of the normals contains as generators the new axes, that is, the lines from $P(x_0, y_0, z_0)$ parallel to the former axes.

To prove this we must note that the equation of a cone of the second degree referred to three of its generators as axes of coordinates is

$$fyz + gzx + hxy = 0. \quad - \quad - \quad - \quad (1)$$

by Example 4, p. 65 of C. Smith's Solid Geometry.

Since equation (6) of the preceding article is of the form (1), the theorem is proved.

5. The cone of the normals C contains as a generator the line from $P(x_0, y_0, z_0)$ to O, the old origin.

The equations of the line OP one are

$$\frac{x}{x_0} = \frac{y}{y_0} = \frac{z}{z_0} \quad - \quad - \quad - \quad - \quad (1)$$

This line OP lies on the cone given by equation (3) of Article 3, since equations (1) satisfy equation (3). It is plain that the origin is a point on the cone C.

6. It is a well known fact proved later on that the cone of the normals, vertex at $P(x_0, y_0, z_0)$, contains the axes of the section obtained by drawing, though the point P, the plane conjugate to the diameter OP which passes through the point P. This fact has served as the point of departure for my succeeding investigations and has enabled me to deduce general formulas for the cone of the normals and also for an allied cone, for the most general central surfaces of the second degree.

7. *Lemma.* Find the equations for the axes of the section of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \quad - \quad - \quad - \quad (1)$$

made by the plane

$$lx+my+nz=0. \quad - \quad - \quad - \quad (2)$$

Let $(x, y, z)$ be the extremity of an axis. If the length of this axis be $r$, we must make $x^2+y^2+z^2=r^2$ a maximum or a minimum, subject to the conditions (1) and (2) above.

Differentiating we have

$$xdx+ydy+zdz=0. \quad - \quad - \quad - \quad (3)$$

$$\frac{x}{a^2}dx+\frac{y}{b^2}dy+\frac{z}{c^2}dz=0. \quad - \quad - \quad - \quad (4)$$

$$ldx+mdy+ndz=0. \quad - \quad - \quad - \quad (5)$$

Hence

$$\begin{vmatrix} x, & y, & z \\ \frac{x}{a^2}, & \frac{y}{b^2}, & \frac{z}{c^2} \\ l, & m, & n \end{vmatrix}=0. \quad - \quad - \quad (6)$$

or

$$\frac{l}{b^2c^2x}(c^2-b^2)+\frac{m}{c^2a^2y}(a^2-c^2)+\frac{n}{a^2b^2z}(b^2-a^2)=0. \quad - \quad (7)$$

This is the equation of a cone having its vertex at the origin. The equations of the axes are then equations (2) and (7). The geometrical interpretation is that the axes are the intersections of the cone given by (7) with the plane given by equation (2).

8. To prove the cone of the normals C, vertex at $P(x_o, y_o, z_o)$, contains the axes of the section obtained by

drawing, through the point P, the plane conjugate to the diameter OP which passes through the point P.

It must be borne in mind that P is the center of the section. This is a well known fact and needs no proof in this thesis.

The plane through the point $P(x_o, y_o, z_o)$, having this point as the center of the section, is

$$lx + my + nz = p. \quad - \quad - \quad - \quad (1)$$

with the condition

$$lx_o + my_o + nz_o = p. \quad - \quad - \quad - \quad (2)$$

Also the diameter conjugate to plane (1), that is, passing through the origin and the point $P(x_o, y_o, z_o)$ the center of the section is

$$\frac{x}{a^2 l} = \frac{y}{b^2 m} = \frac{z}{c^2 n}. \quad - \quad - \quad - \quad (3)$$

Since (3) passes through $P(x_o, y_o, z_o)$, we have the conditions

$$\frac{x_o}{a^2 l} = \frac{y_o}{b^2 m} = \frac{z_o}{c^2 n}. \quad - \quad - \quad - \quad (4)$$

From equations (1) and (2) the plane through P may be written

$$l(x-x_o) + m(y-y_o) + n(z-z_o) = 0,$$

which becomes on employing the conditions (4)

$$\frac{xx_o}{a^2} + \frac{yy_o}{b^2} + \frac{zz_o}{c^2} = \frac{x_o^2}{a^2} + \frac{y_o^2}{b^2} + \frac{z_o^2}{c^2}. \quad - \quad (5)$$

If we take a plane parallel to plane (5), through the center of the ellipsoid, it will have for its equation

$$\frac{xx_o}{a^2} + \frac{yy_o}{b^2} + \frac{zz_o}{c^2} = 0. \quad - \quad - \quad - \quad (6)$$

The axes of the section of the ellipsoid by the plane (6) are given by the intersection of plane (6) with the cone

$$\begin{vmatrix} x, & y, & z \\ \frac{x}{a^2}, & \frac{y}{b^2}, & \frac{z}{c^2} \\ \frac{x_0}{a^2}, & \frac{y_0}{b^2}, & \frac{z_0}{c^2} \end{vmatrix} = 0. \quad - \quad - \quad (7)$$

from equation (6), Article 7, Lemma.

A cone parallel to this, having its vertex at the point $P(x_0, y_0, z_0)$ for origin, will have for its equation the equation (7). But the co-ordinates will be referred to the new axes through P, parallel to the former axes. Transferring back to the original axes by the transformation $x = x - x_0$, $y = y - y_0$, $z = z - z_0$, equation (7) becomes

$$\begin{vmatrix} x-x_0, & y-y_0, & z-z_0 \\ \frac{x-x_0}{a^2}, & \frac{y-y_0}{b^2}, & \frac{z-z_0}{c^2} \\ \frac{x_0}{a^2}, & \frac{y_0}{b^2}, & \frac{z_0}{c^2} \end{vmatrix} = 0.$$

or

$$\begin{vmatrix} x-x_0, & y-y_0, & z-z_0 \\ \frac{x}{a^2}, & \frac{y}{b^2}, & \frac{z}{c^2} \\ \frac{x_0}{a^2}, & \frac{y_0}{b^2}, & \frac{z_0}{c^2} \end{vmatrix} = 0. \quad - \quad - \quad (8)$$

which is the equation of the cone of the normals C by equation (3), Article 3.

The conclusion may be stated as follows:

All parallel sections of an ellipsoid are similar and similarly situated conics. The axes of plane (5) through P are parallel to the axes of plane (6) through the origin. The cones (7) and (8) are parallel. But the cone (7) contains the axes of the section of the ellipsoid by the plane (6). Hence the cone (8) contains the axes of the section of the ellipsoid by plane (5). But the cone (8) is the cone of the normals for the point P. Hence the cone of the normals C, vertex at P, contains the axes of the section obtained by drawing, through the point P, the plane conjugate to the diameter OP, which passes through the point P.

8. A very interesting conclusion follows from this. We have proved in Article 4 that the cone C contains a trirectangular trihedral. If a cone contain one set of three generators mutually at right angles, it contains an infinity of such sets (C. Smith's Solid Geometry, §109.) Since the cone C contains the axes of the section through P (these axes are at right angles), it must also contain the normal through P to the plane of section.

10. By means of the facts so far known we are able to deduce the cone C in a very simple way.

The equation of C referred to axes through its vertex $P(x_0, y_0 z_0)$, parallel to the original axes, is of the form

$$fyz + gzx + hxy = 0.$$

since it contains the new axes as generators. Transforming to the original axes by the transformation $x = x - x_0$, $y = y - y_0$, $z = z - z_0$, this equation becomes

$$f(y-y_0)(z-z_0) + g(z-z_0)(x-x_0) + h(x-x_0)(y-y_0) = 0. \quad (1)$$

Also, since C passes through the origin, $x = y = z = 0$, and equation (1) becomes

$$\frac{f}{x_0} + \frac{g}{y_0} + \frac{h}{z_0} = 0. \quad - \quad - \quad - \quad (2)$$

Since finally it contains the normal

$$\frac{x-x_0}{\left(\frac{x_0}{a^2}\right)}=\frac{y-y_0}{\left(\frac{y_0}{b^2}\right)}=\frac{z-z_0}{\left(\frac{z_0}{c^2}\right)} \quad \text{[Art. 8, Eq. (5)]}$$

to the plane having $P(x_0, y_0, z_0)$ for its center,

$$f\frac{y_0z_0}{b^2c^2}+g\frac{z_0x_0}{c^2a^2}+h\frac{x_0y_0}{a^2b^2}=0. \quad - \quad - \quad (3)$$

Eliminating $f$, $g$ and $h$ between (1), (2) and (3),

$$\begin{vmatrix} (y-y_0)(z-z_0) & (z-z_0)(x-x_0) & (x-x_0)(y-y_0) \\ \frac{1}{x_0} & \frac{1}{y_0} & \frac{1}{z_0} \\ \frac{y_0z_0}{b^2c^2} & \frac{z_0x_0}{c^2a^2} & \frac{x_0y_0}{a^2b^2} \end{vmatrix}=0. \quad (4)$$

This equation, on multiplying the third row by $\frac{a^2b^2c^2}{x_0y_0z_0}$, and dividing the first row by the expression $(x-x_0)(y-y_0)(z-z_0)$, becomes

$$\begin{vmatrix} \frac{1}{x-z_0} & \frac{1}{y-y_0} & \frac{1}{z-z_0} \\ \frac{1}{x_0} & \frac{1}{y_0} & \frac{1}{z_0} \\ \frac{a^2}{x_0} & \frac{b^2}{y_0} & \frac{c^2}{z^2} \end{vmatrix}=0. \quad - \quad (5)$$

or

$$\begin{vmatrix} \frac{x_0}{x-x_0} & \frac{y_0}{y-y_0} & \frac{z_0}{z-z_0} \\ 1 & 1 & 1 \\ a^2 & b^2 & c^2 \end{vmatrix} = 0. \quad - \quad (6$$

on multiplying the first column by $x_0$, the second column by $y_0$, the third column by $z_0$. Equation (6) is the cone of the normals C [See eqation (1), Article 3].

11. I shall next show that the cone of the normals C may be discovered from the equations concerned in finding the axes of the non-central section of an ellipsoid. I was led to this consideration by the fact that the cone C contains the axes of a non-central section, if its vertex is the center of the section.

The center of the section of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

made by the plane

$$lx + my + nz = p \quad - \quad - \quad - \quad (1)$$

is given by the equations

$$\frac{x_0}{a^2 l} = \frac{y_0}{b^2 m} = \frac{z_0}{c^2 n} = \frac{p}{a^2 l^2 + b^2 m^2 + c^2 n^2} \quad - \quad (2)$$

where $P(x_0, y_0, z_0)$ is the center of the section. [See C.Smith's Solid Geometry, Ex. 2, pp. 43-4.] Now transforming the origin to $P(x_0, y_0, z_0)$, with axes through it parallel to the former, by the transformation $x = x + x_0$, $y = y + y_0$, $z = z + z_0$, we see that the equation of the ellipsoid becomes

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) + 2\left(\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2}\right) + \left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} - 1\right) = 0$$

Equation (1) has become, by the transformation above and the condition $lx_0 + my_0 + nz_0 = p$,

$$lx + my + nz = 0.$$

Hence from equations (2),

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = \frac{(lx + my + nz)\,p}{a^2l^2 + b^2m^2 + c^2n^2} = 0.$$

It follows that

$$\left.\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + k = 0 \quad \text{-} \quad \text{-} \quad (3) \\ lx + my + nz = 0 \quad \text{-} \quad \text{-} \quad (4) \end{aligned}\right\}$$

are the equations of the section referred to the new axes, where

$$k = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} - 1.$$

The cone with vertex $P(x_0, y_0, z_0)$ and passing through the intersection of the sphere

$$x^2 + y^2 + z^2 = r^2$$

and the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{c^2}{c^2} + k = 0$$

is given by

$$\frac{x^2}{a^2r^2}(r^2 + ka^2) + \frac{y^2}{b^2r^2}(r^2 + kb^2) + \frac{z^2}{c^2r^2}(r^2 + kc^2) = 0. \quad (5)$$

[See C. Smith's Solid Geometry, Ex. 5, p. 55.]

Every semi-diameter of the surface, whose length is $r$, is a generating line of this cone. This cone will, for all values of

$r$, be cut by the plane (4) in two straight lines which lie along equal diameters of the section; and, when $r$ is equal to *either* semi-axis of the section, these equal diameters will coincide. That is, the plane (4) will touch the cone (5) when $r$ is equal to *either* semi-axes of the section of the ellipsoid by the plane. Hence, to find the equations of the axes, that is, the equations of the lines of contact, we must identify the plane (4) wifh the tangent plane to (5) at some point $(x_1, y_1, z_1)$. This tangent plane may be written

$$\frac{xx_1}{a^2r^2}(r^2 + ka^2) + \frac{yy_1}{b^2r^2}(r^2 + kb^2) + \frac{zz_1}{c^2r^2}(r^2 + kc^2) = 0.$$

which, when compared with (4), gives the equations

$$\frac{\frac{x_1}{a^2r^2}(r^2 + ka^2)}{l} = \frac{\frac{y_1}{b^2r^2}(r^2 + kb^2)}{m} = \frac{\frac{z_1}{c^2r^2}(r^2 + kc^2)}{n}$$

Putting each of these fractions equal to $-\lambda$, we have, on dropping accents,

$$\frac{x}{a^2} + x\frac{k}{r^2} + \lambda l = 0.$$

$$\frac{y}{b^2} + y\frac{k}{r^2} + \lambda m = 0.$$

$$\frac{z}{c^2} + z\frac{k}{r^2} + \lambda n = 0.$$

Eliminating $\frac{k}{r^2}$ and $\lambda$, we obtain

$$\begin{vmatrix} \frac{x}{a^2} & x & l \\ \frac{y}{b^2} & y & m \\ \frac{z}{c^2} & z & n \end{vmatrix} = 0.$$

Transforming to the original axes by the transformation $x = x - x_0$, $y = y - y_0$, $z = z - z_0$, we have

$$\begin{vmatrix} \frac{x - x_0}{a^2}, & x - x_0, & l \\ \frac{y - y_0}{b^2}, & y - y_0, & m \\ \frac{z - z_0}{c^2}, & z - z_0, & n \end{vmatrix} = 0.$$

which becomes on applying equations (2)

$$\begin{vmatrix} \frac{x - x_0}{a^2}, & x - x_0, & \frac{x_0}{a^2} \\ \frac{y - y_0}{b^2}, & y - y_0, & \frac{y_0}{b^2} \\ \frac{z - z_0}{c^2}, & z - z_0, & \frac{z_0}{c^2} \end{vmatrix} = 0.$$

which is the cone of the normals for the point $P(x_0, y_0, z_0)$ by equation (2), Article 3.

The geometrical interpretation is that the axes are obtained by the intersection of the cone of the normals for $P(x_o, y_o, z_o)$ with the plane

$$lx + my + nz = p.$$

12. To prove geometrically that the cone of the normals for P contains the line OP from P to the origin.

By Article 7, Lemma, the cone having its vertex at the origin, parallel to the cone of the normals, and containing the axes of the section of $lx + my + nz = 0$, and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, is given by

$$\begin{vmatrix} x & y & z \\ \frac{x}{a^2} & \frac{y}{b^2} & \frac{z}{c^2} \\ l & m & n \end{vmatrix} = 0. \quad - \quad - \quad (1)$$

Since this cone is of the form

$$fyz + gzx + hxy = 0,$$

it contains as generators three conjugate diameters of the ellipsoid. [C. Smith's Solid Geometry, Ex. 5, p. 90.] Since the diameter conjugate to the plane $lx + my + nz = 0$ - (2) forms, with the axes of the section, three conjugate diameters, therefore the cone (1) contains the diameter conjugate to plane (2), that is the line OP.

Hence the cone of the normals for $P(x_o, y_o, z_o)$, parallel to and exactly like cone (1), having one point in common with a generator of (1) must contain that generator, which is OP (Compare Article 5).

13. I am seeking the general equation of the cone of the normals for central surfaces of the second degree, given in their most general form. Since the method of Article 11 is

perfectly general in its application, I shall apply it to the general equation of the second degree.

The general equation of the second degree having its center at the origin is

$$f(x, y, z) = ax^2+by^2+cz^2+2fyz+2gzx+2hxy+d' = \phi(x, y, z)+d' = 0. \quad - \quad - \quad (1)$$

[See C. Smith's Solid Geometry, Article 75.]

Transform to parallel axes through the center $P(x_o, y_o, z_o)$ of the section of $f(x, y, z) = 0$ by the plane

$$lx + my + nz = p \quad - \quad - \quad - \quad (2)$$

by means of the transformation equations $x = x+x_o$, $y = y+y_o$ $z = z + z_o$.

We must note, however, that the center $P(x_o, y_o, z_o)$ of the section is given by

$$\frac{1}{l}\frac{\delta f}{\delta x_o} = \frac{1}{m}\frac{\delta f}{\delta y_o} = \frac{1}{n}\frac{\delta f}{\delta z_o} = -2\lambda \text{ say.} \quad - \quad (3)$$

[Compare C. Smith's Solid Geometry, Ex. 2, pages 43 and 44.] That is

$$\left.\begin{aligned} ax_o + hy_o + gz_o + \lambda l = 0, \\ hx_o + by_o + fz_o + \lambda m = 0, \\ gx_o + fy_o + cz_o + \lambda n = 0, \\ lx_o + my_o + nz_o - p = 0. \end{aligned}\right\} \quad - \quad - \quad (4)$$

By the transformation mentioned above $f(x, y, z) = 0$ becomes

$$\phi(x, y, z) + f(x_o, y_o, z_o) + 2axx_o + 2byy_o + 2czz_o + 2f(yz_o + y_oz) + 2g(zx_o + z_ox) + 2h(xy_o + x_oy) = 0. \quad (5)$$

or $$\phi(x, y, z) + f(x_o, y_o, z_o) + 2x(ax_o + hy_o + gz_o) + 2y(hx_o + by_o + fz_o) + 2z(gx_o + fy_o + cz_o) = 0. \quad (6)$$

On applying formulas (4), equation (6) becomes

$$\phi(x, y, z) + f(x_0, y_0, z_0) - 2\lambda(lx + my + nz) = 0. \quad (7)$$

But since $lx + my + nz = p$ and $lx_0 + my_0 + nz_0 = p$, where $x, y, z$ are referred to old origin,

$$l(x - x_0) + m(y - y_0) + n(z - z_0) = 0$$

and since too $x = x + x_0$, $y = y + y_0$, $z = z + z_0$, we have

$$lx + my + nz = 0, \quad - \quad - \quad - \quad (8)$$

the equation of the plane through $P(x_0, y_0, z_0)$, the new origin.

Hence, from (7),

$$\left.\begin{aligned} \phi(x, y, z) + k &= 0. \\ lx + my + nz &= 0. \end{aligned}\right\}$$

are the equations of the section, where

$$k = f(x_0, y_0, z_0).$$

The cone with vertex at P and passing through the intersection of the sphere

$$x^2 + y^2 + z^2 = r^2$$

and the surface

$$\phi(x, y, z) + k = 0$$

is given by

$$r^2\phi(x, y, z) + k(x^2 + y^2 + z^2) = 0 \quad - \quad (9)$$

which is obtained by forming a homogeneons equation from these two equations.

Every semi-diameter of the surface, whose length is $r$, is a generating line of cone (9). This cone will, for all values of $r$, be cut by the plane $lx + my + nz = 0$ in two straight lines which lie along equal diameters of the section, and,

when $r$ is equal to either semi-axis of the section, these equal diameters will coincide. That is, the plane $lx + my + nz = 0$ will touch the cone when $r$ is equal to *either* semi-axis of the section of the surface by the plane. Hence, to find the equations of the axes, that is, the equations of the lines of contact, we must identify the plane $lx + my + nz = 0$ with the tangent plane to the cone at some point $(x_1, y_1, z_1)$. This tangent plane is

$$(r^2\phi'_{x_1} + 2kx_1)\,x + (r^2\phi'_{y_1} + 2ky_1)\,y + (r^2\phi'_{z_1} + 2kz_1)\,z = 0.$$

Hence

$$\frac{r^2\phi'_{x_1} + 2kx_1}{l} = \frac{r^2\phi'_{y_1} + 2ky_1}{m} = \frac{r^2\phi'_{z_1} + 2kz_1}{n}.$$

Then

$$\frac{\phi'_{x_1} + 2\frac{k}{r^2}x_1}{l} = \frac{\phi'_{y_1} + 2\frac{k}{r^2}y_1}{m} = \frac{\phi'_{z_1} + 2\frac{k}{r^2}z_1}{n} = 2\lambda' \text{ say.}$$

Clearing out and dropping accents,

$$\phi'_x + \frac{2k}{r^2}x - 2\lambda' l = 0$$

$$\phi'_y + \frac{2k}{r^2}y - 2\lambda' m = 0$$

$$\phi'_z + \frac{2k}{r^2}z - 2\lambda' n = 0$$

Hence

$$\begin{vmatrix} \phi'_x & x & l \\ \phi'_y & y & m \\ \phi'_z & z & n \end{vmatrix} = 0.$$

Transform back to the original axes by the substitution $x = x - x_0$, $y = y = y_0$, $z = z - z_0$, and note that

$$\begin{aligned}\phi'_{x-x_0} &= 2(ax + hy + gz)]_{x-x_0} \\ &= 2[(ax + hy + gz) - (ax_0 + hy_0 + gz_0)] \\ &= f'_x - f'_{x_0} = f'_x + 2\lambda l \text{ from equations (4).}\end{aligned}$$

We have then

$$\begin{vmatrix} \frac{\delta f}{\delta x} + 2\lambda l & x - x_0 & l \\ \frac{\delta f}{\delta y} + 2\lambda m & y - y_0 & m \\ \frac{\delta f}{\delta z} + 2\lambda n & z - z_0 & n \end{vmatrix} = 0.$$

or

$$\begin{vmatrix} \frac{\delta f}{\delta x} & x - x_0 & l \\ \frac{\delta f}{\delta y} & y - y_0 & m \\ \frac{\delta f}{\delta z} & z - z_0 & n \end{vmatrix} = 0. \quad (10)$$

which becomes by virtue of the equations

$$\frac{1}{l}\frac{\delta f}{\delta x_0} = \frac{1}{m}\frac{\delta f}{\delta y_0} = \frac{1}{n}\frac{\delta f}{\delta z_0}$$

$$\begin{vmatrix} \frac{\delta f}{\delta x} & x - x_0 & \frac{\delta f}{\delta x_0} \\ \frac{\delta f}{\delta y} & y - y_0 & \frac{\delta f}{\delta y_0} \\ \frac{\delta f}{\delta z} & z - z_0 & \frac{\delta f}{\delta z_0} \end{vmatrix} = 0. \quad - \quad (11)$$

The equation (11), by analogy with the result of Article 11, should prove to be the cone of the normals for the point P.

We shall now apply the tests, which are of the following nature:

I. It should contain the normal to the plane $lx+my+nz=p$, drawn through the point $P(x_0, y_0, z_0)$.

II. It should pass through the origin.

III. It should be capable of containing three generators at right angles.

Since equation (10) is satisfied for

$$\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n},$$

the equations of the normal to the plane

$$lx + my + nz = p$$

through the point $P(x_0, y_0, z_0)$, condition I is satisfied.

Test II is verified since equation (11) is satisfied for $x = y = z = 0$.

Test III is verified since the cone (11) contains the normal, to the plane $lx + my\ nz = p$, drawn through the point $P(x_0, y_0, z_0)$ and also, by the method of derivation, contains the axes of the section of the conicoid by the plane $lx + my + nz = p$.

Hence equation (11) is the equation of the cone of the nor-

mals for the point $P(x_0, y_0, z_0)$ for the most general central surface of the second degree.

14. A second proof may be obtained as follows: The equations

$$\left.\begin{aligned} \phi(x, y, z) + k = 0 \\ lx + my + nz = 0 \end{aligned}\right\} \quad - \quad - \quad - \quad (1)$$

of the preceding article suffice to define the section whose axes we are seeking.

Let $(x, y, z)$ be the co-ordinates of the extremity of an axis whose length is $r$.

Hence

$$x^2 + y^2 + z^2 = r^2$$

must be a maximum or a minimum, subject to the conditions (1).

Therefore

$$\phi'_x\, dx + \phi'_y\, dy + \phi'_z\, dz = 0$$

$$x\, dx + y\, dy + z\, dz = 0$$

$$l\, dx + m\, dy + n\, dz = 0$$

Hence

$$\begin{vmatrix} \phi'_x & \phi'_y & \phi'_z \\ x & y & z \\ l & m & n \end{vmatrix} = 0.$$

This is the same as equation deduced in Article 13. The remainder of the proof follows precisely as in Article 13.

15. Equation (11) when applied to the equation of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

gives the result already found in Article 3, which is a further demonstration that it is the true equation desired. It may also be applied to the equation of the hyberboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

and to the equation of the hyperboloid of two sheets

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

Equation (11), Article 13, is the equation of the cone of the normals for the following reasons: There are three central surfaces of the second degree—the ellipsoid and the hyperboloids of one and two sheets. The three tests applied in Article 13 are properties belonging to any central surface of the second degree, since the cone of the normals for the ellipsoid may be immediately transformed into the corresponding equations for the hyperboloids of one and two sheets by changing $c^2$ into $-c^2$, and $b^2$ into $-b^2$, $c^2$ into $-c^2$ respectively. Hence the properties proved for the ellipsoid hold also for the hyperboloids and hence for all central surfaces of the second degree.

16. The feet of the normals from a point $P(x_o, y_o, z_o)$ to the ellipsoid lie upon a cone of the second degree whose vertex is at the origin.

The equations of the normal to the ellipsoid at the point $(f, g, h)$ are

$$\frac{x-f}{\left(\frac{f}{a^2}\right)}=\frac{y-g}{\left(\frac{g}{b^2}\right)}=\frac{z-h}{\left(\frac{h}{c^2}\right)}=\lambda \text{ say.}$$

Since this normal passes through the point $P(x_o, y_o, z_o)$, we have

$$\frac{x_0 - f}{\left(\frac{f}{a^2}\right)} = \frac{y_0 - g}{\left(\frac{g}{b^2}\right)} = \frac{z_0 - h}{\left(\frac{h}{c^2}\right)} = \lambda' \text{ say.}$$

Hence $x_0 - f = \frac{f}{x}\lambda'$, or $a^2 + \lambda' = \frac{a^2 x_0}{f}$.

Then

$$(a^2 + \lambda') = \frac{a^2 x_0}{x}, \quad (b^2 + \lambda') = \frac{b^2 y_0}{y}, \quad (c^2 + \lambda') = \frac{c^2 z_0}{z},$$

where $(x, y, z)$ as *current co-ordinates* replace $(f, g, h)$.

Multiplying these equations respectively by $(b^2 - c^2)$, $(c^2 - a^2)$, $(a^2 - b^2)$ and adding we obtain

$$\frac{a^2 x_0 (b^2 - c^2)}{x} + \frac{b^2 y_0 (c^2 - a^2)}{y} + \frac{c^2 z_0 (a^2 - b^2)}{z} = 0. \qquad (C')$$

This is the equation of a cone, vertex at the origin, and passing through the feet of the normals from P $(x_0, y_0, z_0)$ to the ellipsoid.

This cone passes through P $(x_0, y_0, z_0)$, since it is satisfied for $x = x_0$, $y = y_0$, $z = z_0$, and hence the line joining the origin to the point P is a generator of this cone.

The intersection of the two cones C and C′ is composed:—

(*i*) of the straight line OP,

(*ii*) of a twisted curve which we shall designate by the letter K.

The intersection of K with the ellipsoid gives the six points which are the feet of the six normals from the point P $(x_0, y_0, z_0)$ to the ellipsoid.

Since the cone C′ is of the form

$$Ayz + Bzx + Cxy = 0$$

it contains three perpendicular generators. [This property

holds for any central surface of the second degree, since C′ retains the same form when $b^2$ and $c^2$ are changed into $-b^2$ and $-c^2$.]

[Compare C. Smith's Solid Geometry, Article 109, p. 85.]

If it contains one set, it sontains an infinity of such sets. A new method at once suggests itself for deducing the equation of cone C′. As my purpose has been to deduce the equation of the cone C′ for the most general equation of a central surface of the second degree, I was foiled up to this point since the method of solution above given is not general in its application. The new method which I shall presently give is perfectly general in its application and has enabled me to deduce the equation of the cone C′ for the most general equation of central surfaces of the second degree.

17. The line from the origin to the point P is the normal to the central plane

$$x_0x + y_0y + z_0z = 0. \quad - \quad - \quad - \quad (1)$$

Since the cone C′ contains any set of three perpendicular generators, it contains the normal to the plane (1) and should contain the axes of the section of the ellipsoid by the plane (1). Now, by Article 7 above, the cone giving by its intersection with (1) the axes of the section is

$$\frac{x_0}{b^2c^2x}(c^2 - b^2) + \frac{y_0}{c^2a^2y}(a^2 - c^2) + \frac{z_0}{a^2b^2z}(b^2 - a^2) = 0,$$

passing through O and P, or

$$\frac{a^2x_0}{x}(b^2 - c^2) + \frac{b^2y_0}{y}(c^2 - a^2) + \frac{c^2z_0}{z}(a^2 - b^2) = 0,$$

which is the cone C′ as found in Article 16.

I shall next apply this method to the general equation of the second degree having its center at the origin.

18. Find the equations for the axes of the section of the surface

$$f(x, y, z) = ax^2+by^2+cz^2+2fyz+2gzx+2hxy+d' = 0 \quad (1)$$

made by the plane

$$lx + my + nz = 0. \quad - \quad - \quad - \quad (2)$$

Let $(x, y, z)$ be the co-ordinates of the extremity of an axis. If the length of this axis be $r$, we must make

$$x^2 + y^2 + z^2 = r^2$$

a maximum or a minimum, subject to the conditions (1) and (2).

Hence

$$\left.\begin{aligned} \frac{\delta f}{\delta x}dx + \frac{\delta f}{\delta y}dy + \frac{\delta f}{\delta z}dz &= 0 \\ x\,dx + y\,dy + z\,dz &= 0 \\ l\,dx + m\,dy + n\,dz &= 0 \end{aligned}\right\}$$

Then

$$\begin{vmatrix} \frac{\delta f}{\delta x} & \frac{\delta f}{\delta y} & \frac{\delta f}{\delta z} \\ x & y & z \\ l & m & n \end{vmatrix} = 0. \quad - \quad - \quad (3)$$

is the equation of the cone which gives by its intersection with $lx + my + nz = 0$ the equation of the axes of the section.

19. Find the equation of a cone whose vertex is at the origin and which passes through the feet of the normals from $P(x_0, y_0, z_0)$ to the surface

$$f(x, y, z) = ax^2+by^2+cz^2+2fyz+2gzx+2hxy+d' = 0. \quad (1)$$

We know by Article 17 that this cone determines by its in-

tersection with the plane through the origin perpendicular to OP

$$x_0x + y_0y + z_0z = 0 \quad - \quad - \quad - \quad (2)$$

the axes of the section of (1) by (2).

By equation (3), Article 18, this cone has for its equation

$$\begin{vmatrix} \frac{\delta f}{\delta x} & \frac{\delta f}{\delta y} & \frac{\delta f}{\delta z} \\ x & y & z \\ x_0 & y_0 & z_0 \end{vmatrix} = 0. \quad - \quad - \quad (3)$$

Is this the required cone C'? Two tests will now be applied. In order that equation (3) should represent the equation of Cone C', it should contain as a generator the line OP. This is true, since equation (3) is satisfied for $x = x_0$, $y = y_0$, $z = z_0$.

It should also be capable of containing three generators at right angles.

Now the condition that a cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

shall contain three generators mutually at right angles is

$$a + b + c = 0. \quad - \quad - \quad - \quad (4)$$

[C. Smith's Solid Geometry, §109, page 85.]

Now the cone (3), when cleared out, becomes

$$(gy_0 - hz_0)x^2 + (hz_0 - fx_0)y^2 + (fx_0 - gy_0)z^2 + \ldots = 0.$$

Applying condition (4) we have

$$(gy_0 - hz_0) + (hz_0 - fx_0) + (fx_0 - gy_0) = 0,$$

which is identically true.

A further verification of the truth of this result is made as follows:

The equation (3) when applied to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

gives for the equation of C'

$$\begin{vmatrix} \frac{x}{a^2} & \frac{y}{b^2} & \frac{z}{c^2} \\ x & y & z \\ x_0 & y_0 & z_0 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} \frac{1}{a^2} & \frac{1}{b^2} & \frac{1}{c^2} \\ 1 & 1 & 1 \\ \frac{x_0}{x} & \frac{y_0}{y} & \frac{z_0}{z} \end{vmatrix} = 0$$

or

$$\frac{a^2x_0}{x}(b^2-c^2)+\frac{b^2y_0}{y}(c^2-a^2)+\frac{c^2z_0}{z}(a^2-b^2)=0.$$

The last equation written is identical with equation (C') of Article 16, thus completing our verification of this equation (3).

In like manner the cone C' for the hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

may be written down

$$\begin{vmatrix} \frac{x}{a^2} & \frac{y}{b^2} & \frac{z}{-c^2} \\ x & y & z \\ x_0 & y_0 & z_0 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} \frac{1}{a^2} & \frac{1}{b^2} & \frac{1}{-c^2} \\ 1 & 1 & 1 \\ \frac{x_0}{x} & \frac{y_0}{y} & \frac{z_0}{z} \end{vmatrix} = 0$$

or

$$\frac{a^2x_0}{x}(b^2+c^2) - \frac{b^2y_0}{y}(c^2+a^2) + \frac{c^2z_0}{z}(b^2-a^2) = 0.$$

Lastly, the cone C′ for the hyperboloid of two sheets

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

is given by the equation

$$\begin{vmatrix} \frac{x}{a^2} & \frac{y}{-b^2} & \frac{z}{-c^2} \\ x & y & z \\ x_0 & y_0 & z_0 \end{vmatrix} = 0$$

or

$$\begin{vmatrix} \frac{1}{a^2} & -\frac{1}{b^2} & -\frac{1}{c^2} \\ 1 & 1 & 1 \\ \frac{x_0}{x} & \frac{y_0}{y} & \frac{z_0}{z} \end{vmatrix} = 0$$

or

$$\frac{a^2x_0}{x}(c^2 - b^2) + \frac{b^2y_0}{y}(c^2 + a^2) - \frac{c^2z_0}{z}(a^2 + b^2) = 0$$